SQlsign2D Dimensional Goldilocks

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SQlsign2D

- A post-quantum digital signature protocol
- As compact as SQIsign
- As safe as SQIsign8D (safer than SQIsign4D, SQIsign)
- The fastest verification of all SQIsign protocols
- Aim is to update the SQIsign NIST submission to 2D

East vs West vs Prime

- Within a few days, three variants of SQISign were published
- Today I'll be talking about **SQIsign2D-West**
- SQIsign2D-East is very similar to West, but with heuristics and faster signing
- SQIsignPrime is more similar to SQIsign4D with a new challenge
- Neither East or Prime has an implementation





	Sizes (Bytes)		Timing (ms)		
	Public Key	Signature	Keygen	Sign	Verify
NIST I	66	148	30	80	4.5
NIST III	98	222	85	230	14.5
NIST V	130	294	180	470	31





Very compact signatures

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Isogenists: very fast algorithms?!

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SQIsign2D



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Slow algorithms

SQIsign2D



Can SQlsign2D be fast enough when size-restrictions force its use?



- Understanding SQIsign signatures
- Why is dimension two "just right"
- How does SQlsign2D compare
- What next?

Isogenies and Friends

sogeny World

- Elliptic curves are curves: maps between curves are rational maps
- Elliptic curves are groups: maps between groups are homomorphisms
- An isogeny $\varphi: E_1 \to E_2$ is a map between curves which additionally preserves the group structure

ring End(E)

$\varphi(P+Q) = \varphi(P) + \varphi(Q), \quad P, Q \in E_1$

• An isogeny $\theta: E \to E$ is an endomorphism, the set of endomorphisms is a

Supersingular Isogeny World

- Supersingular curves have particularly large endomorphism rings
- For the curves we consider: E/\mathbb{F}_{p^2} , End(E) has rank four
- Isogenies have finite kernels and we're interested in separable isogenies: $\#ker(\varphi) = deg(\varphi)$
- For efficiency, we generally can only compute smooth degree isogenies
- For a given ℓ , we can compute the ℓ -isogeny graph which are ($\ell + 1$) regular and Ramanujan (it's easy to get lost in the graph)

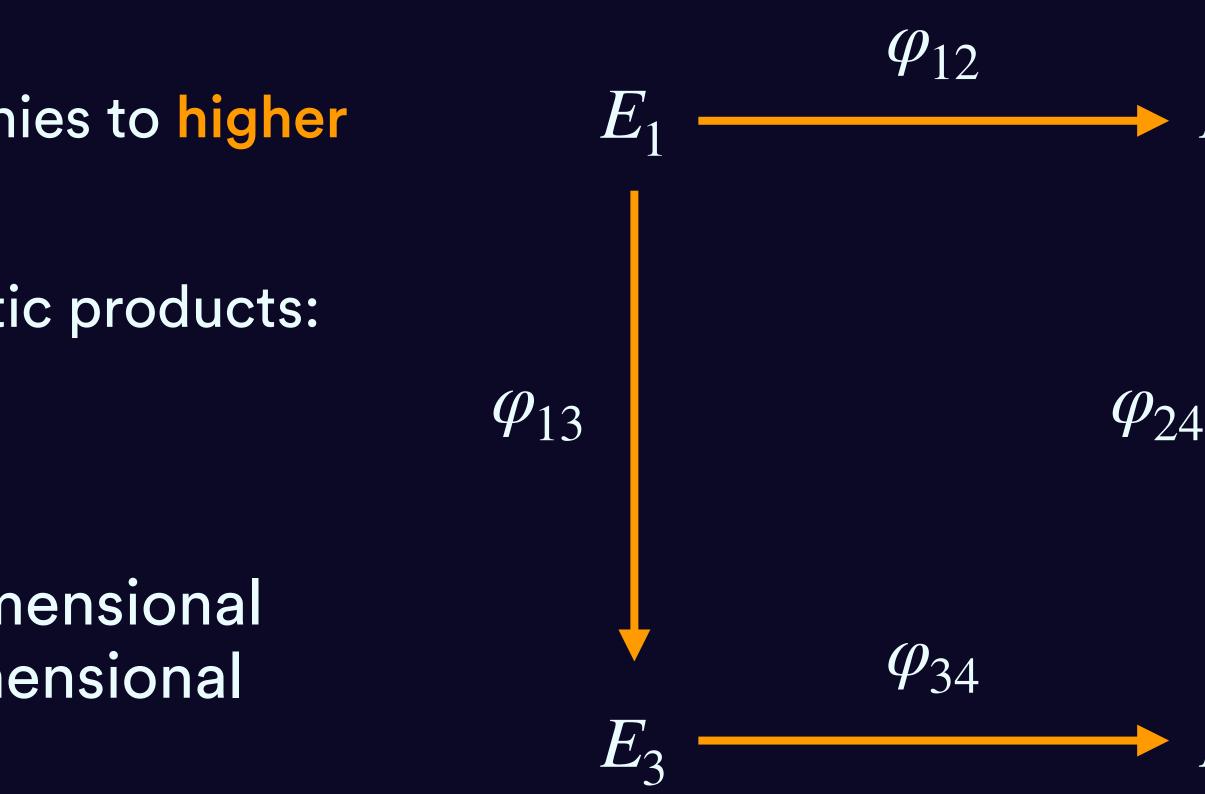
HD Isogeny World

- We can generalise the notion of isogenies to higher dimensional varieties
- A trendy isogeny is one between elliptic products:

$$\Phi: E_1 \times E_4 \to E_2 \times E_3$$

 We can think of this as one two-dimensional isogeny or a matrix of four one-dimensional isogenies:

$$\Phi = \begin{pmatrix} \varphi_{12} & \widehat{\varphi}_{34} \\ -\varphi_{13} & \widehat{\varphi}_{24} \end{pmatrix}$$







Quaternion World

- An "extension" of complex numbers, elements look like: $\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ • For $\alpha \in \mathscr{B}_{p,\infty} = \mathbb{Q}\langle i,j \rangle$ we have $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = -p$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$ • Given a fractional ideal I, the left order is $\mathcal{O}_L(I) = \{ \alpha \in \mathscr{B}_{p,\infty} \mid \alpha I \subset I \}$ • Why quaternions? When End(E) has rank four, $End(E) \cong O$ (maximal) • We'll discuss this more Deuring the talk...

Hard Problems Or at least not easy yet...

- Given two supersingular elliptic curves, find an isogeny connecting them
- Given a supersingular elliptic curve, compute its endomorphism ring
- In (2021/919) Wesolowski showed these problems are equivalent
- There are other "hard" isogeny problems, some of which are now understood to be easy
- Given an isogeny-based cryptographic primitive, convince people that it's practical

Digital Signatures

I know something you don't know and I can prove it to you.



I know the...

Endomorphism ring of this supersingular curve



Public set-up

- A prime p

- A supersingular elliptic curve E_0/\mathbb{F}_{p^2} with known endomorphism ring $\mathcal{O}_0 \cong \operatorname{End}(E_0)$

$$E_0: y^2 = x^3 + x \qquad (p \equiv 3 \mod 4)$$



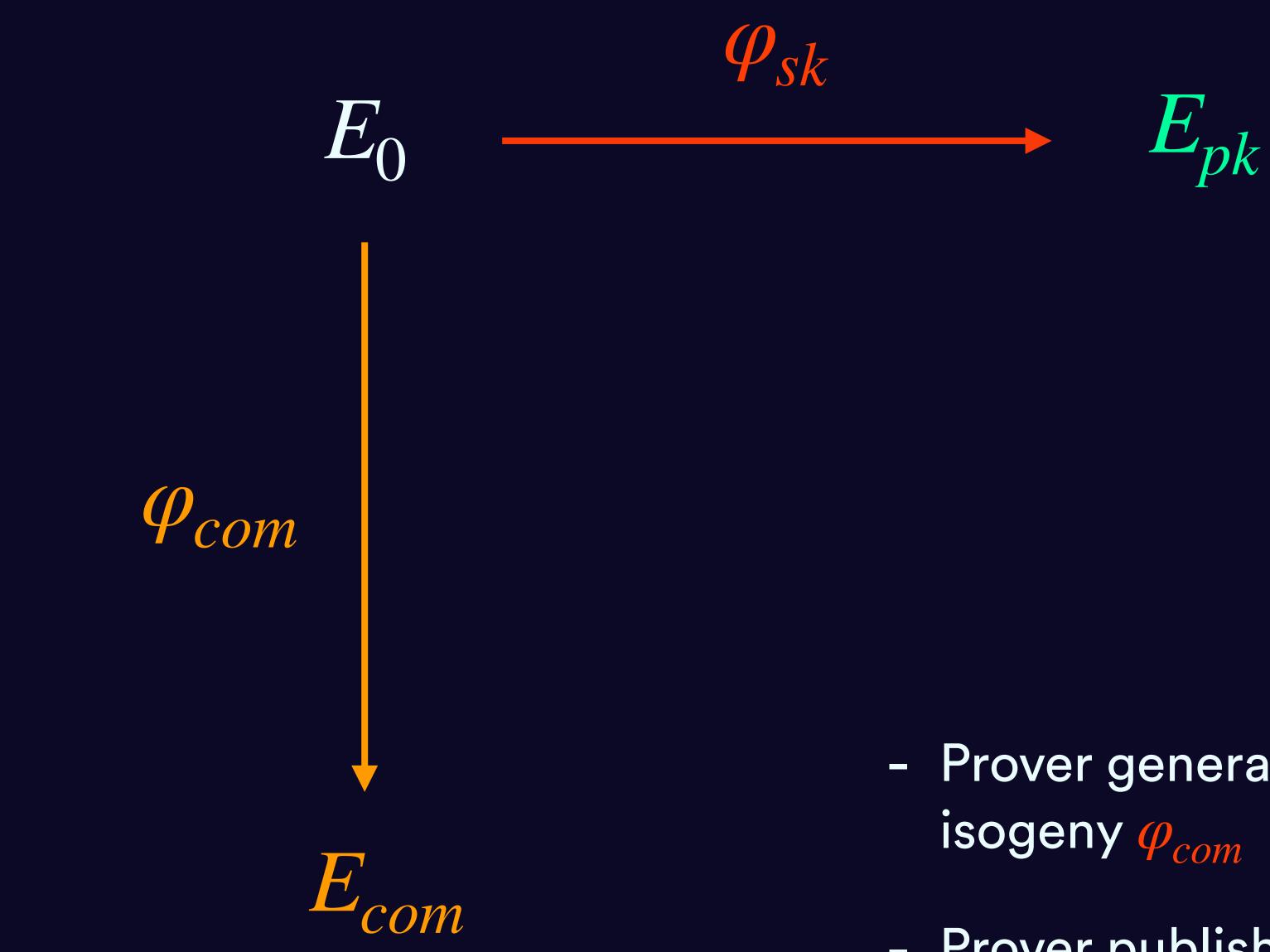






Keygen

- Prover generates a secret random isogeny φ_{sk}
- Prover publishes the codomain E_{pk}



Commitment

- Prover generates a secret random isogeny φ_{com}
- Prover publishes the codomain E_{com}





Verifier generates a public,

 $\varphi_{chl}: E_{pk} \to E_{chl}$

Ecom

 E_0

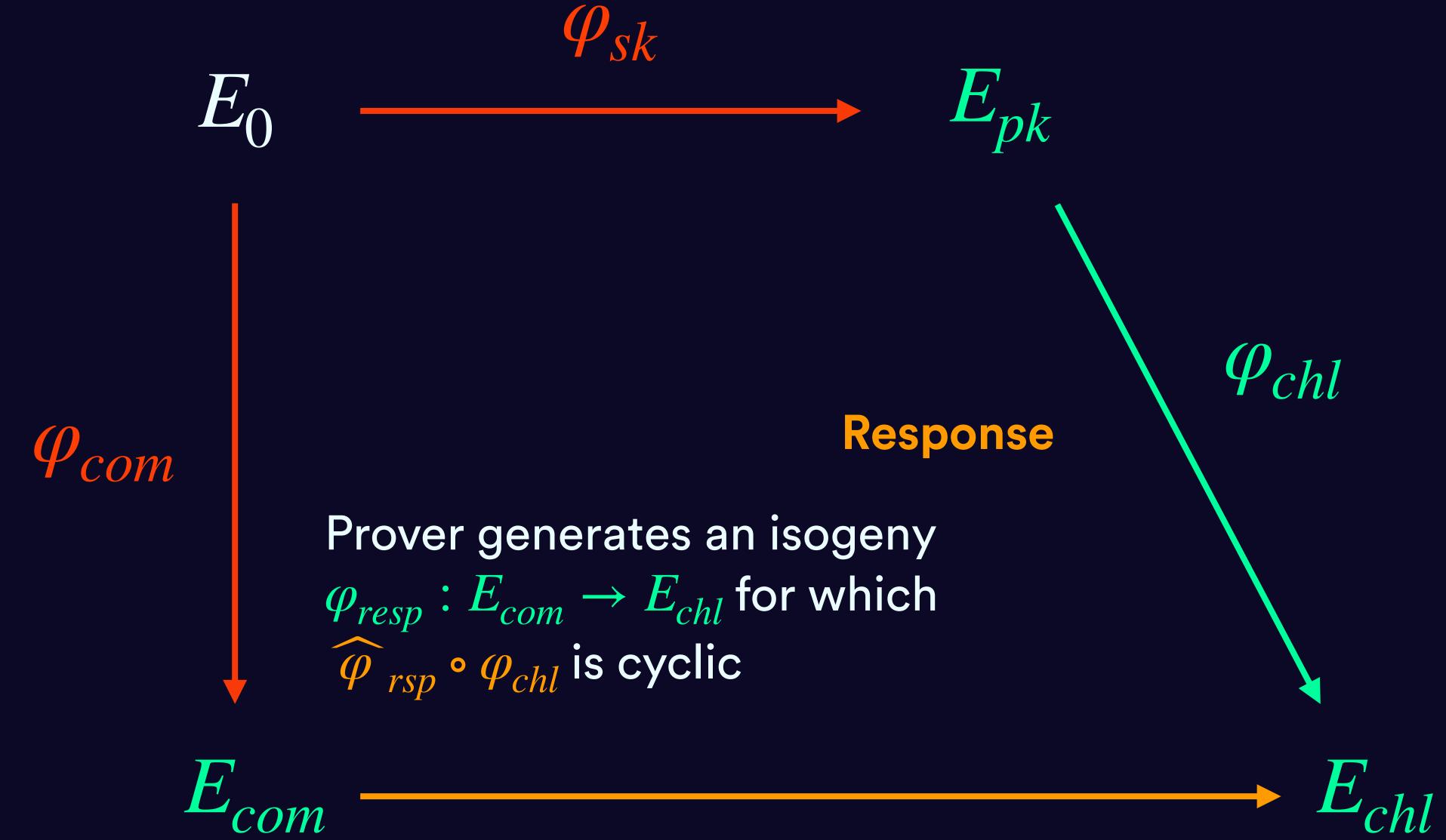
Challenge

 E'_{pk}

random, smooth degree isogeny

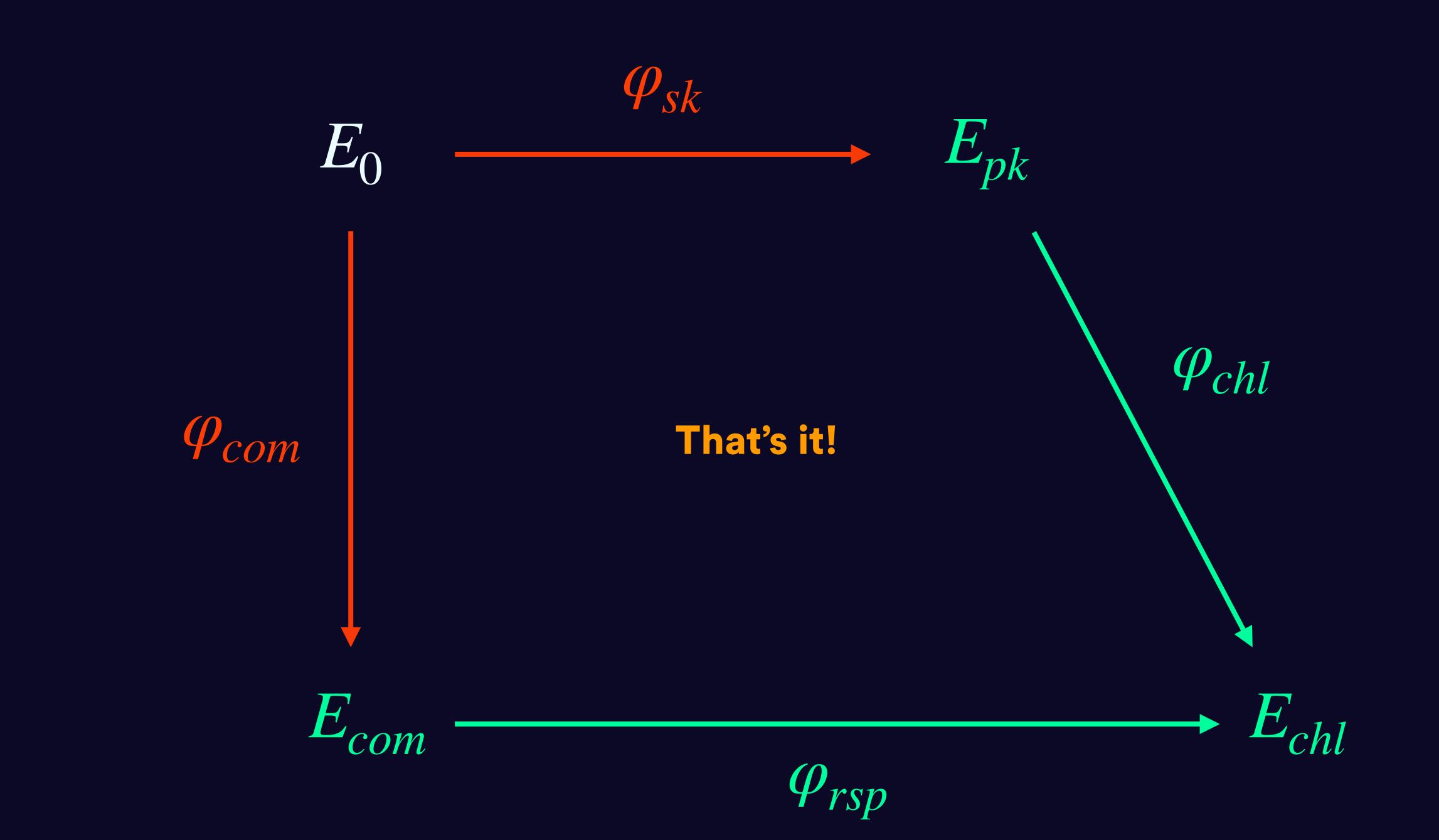


 φ_{chl}









Signatures from Identification Protocols

- We use the Fiat-Shamir transform to obtain an asymmetric signature
- Interactive challenge is replaced by a deterministic challenge
- Commitment recoverability reduces signature size
- Verification of a signature is ensuring response validity

Where's the endomorphism ring?

The Deuring Correspondence

- Deuring showed that there's equivalences between the isogeny world and quaternion world
- In supersingular isogeny world, we have curves, points on curves and isogenies between curves
- In quaternion world we have (maximal) orders, quaternions and ideals
- Importantly, hard isogeny problems can be easy quaternion problems
- The dictionary we need is the endomorphism ring of a curve

Some Cool Facts Impress your friends at dinner!

- orders
- If two ideals are equivalent then the corresponding isogenies have isomorphic (co)domains
- Endomorphisms are equivalent to principle ideals
- The dual isogeny is equivalent to the conjugate ideal
- An isogeny of degree d is equivalent to an ideal of norm n

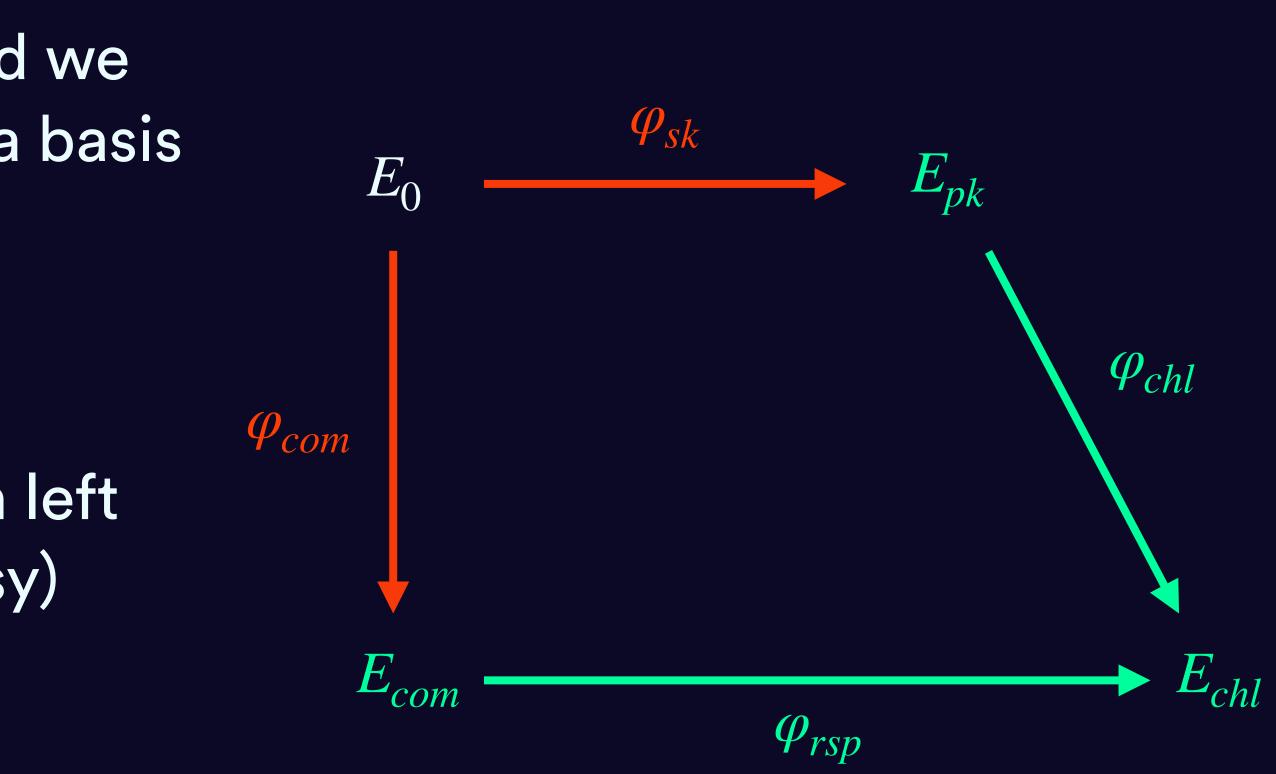
• An isogeny connecting two curves is equivalent to an ideal connecting two

Easy Problems

- Given End(E) compute the corresponding maximal order
- Given two maximal orders, compute the connecting ideal
- KLPT: Given an ideal compute an equivalent ideal with smooth norm
- Given an ideal compute the corresponding isogeny

Calculating the Response

- As we know (a basis) of $End(E_0)$ and we know φ_{com} and $\varphi_{sk} \circ \varphi_{chl}$, we know a basis of End($\overline{E_{com}}$) and End($\overline{E_{chl}}$)
- We then know \mathcal{O}_{com} and \mathcal{O}_{chl}
- Compute the connecting ideal with left order \mathcal{O}_{com} and right order \mathcal{O}_{chl} (easy)
- This ideal is equivalent to φ_{rsp}
- Only possible because we know secrets!

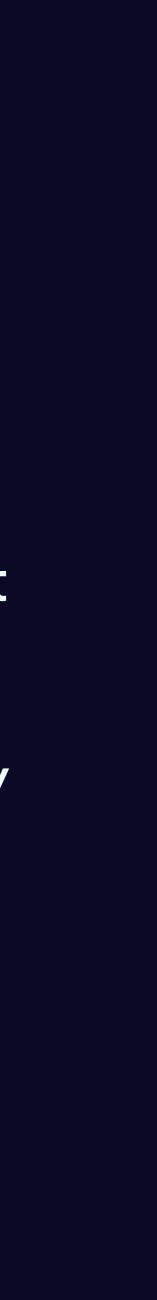


SQIsign in dimensions 1, 2, 4, ... Dimension One

- For dimension one the response was computed using the KLPT algorithm
- Practical concerns restricted suitable characteristics, which meant higher security levels were hard to find parameters for
- The response (2ⁿ)-isogeny was very long: with $\deg(\varphi_{rsp}) = p^{15/4}$
- This meant the isogeny was computed in many smaller steps (slow!)
- Recent work (ApresSQI) tries to speed up verification by allowing larger step sizes

SQIsign in dimensions 1, 2, 4, 8 **Dimension Four and Eight**

- For SQIsignHD the response isogeny no longer needed to be smooth
- If you work in dimension eight, we get great security proofs, but no one has yet implemented dimension eight isogenies
- If you work in dimension four, you introduce heuristics and weaken the security proof
- Resulting protocol was even more compact than SQIsign
- Signing was much faster than SQIsign and parameters easy to choose
- Verification needed (2ⁿ, 2ⁿ, 2ⁿ, 2ⁿ)-isogenies which are slow



SQIsign in dimensions 1, 2, 4, 8 Dimension Two

- In SQIsign2D we obtain the same strong security guarantees of rigorous SQIsign8D but with "fast" 2D isogenies
- Signatures are less compact than SQIsign4D but more compact that SQIsign
- Signing is much faster than SQIsign and but slower than SQIsign4D
- Verification is the fastest of all variants (potentially even ApresSQI)





Evaluate a random isogeny of given degree



Translate an ideal into a twodimensional isogeny

Fixed Degree sogeny

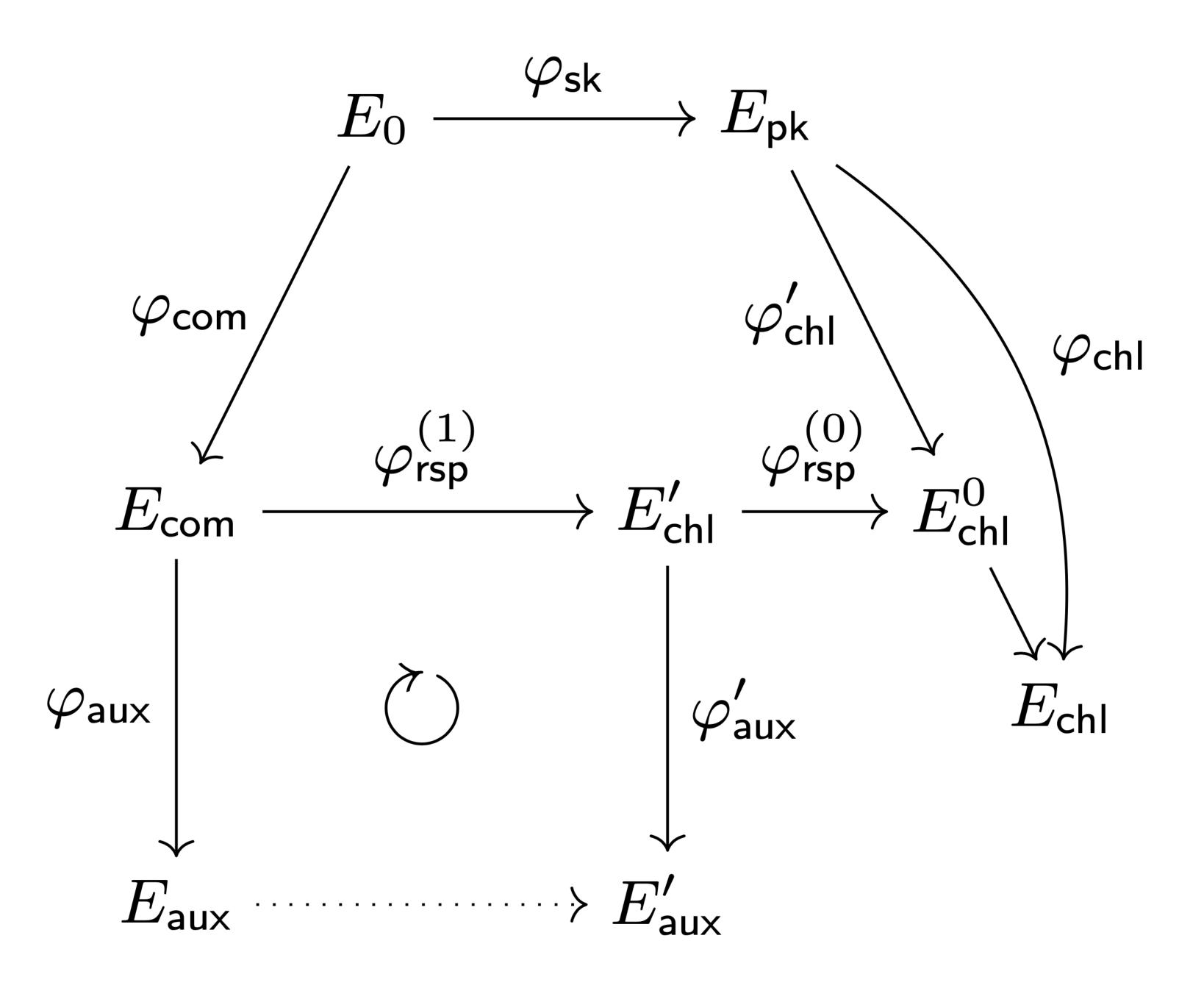
- Input: a curve with known $End(E_0)$, a basis $E[2^e] = \langle P, Q \rangle$ and odd integer $u < 2^e$
- E' and the corresponding ideal
 - Sample $\theta \in \text{End}(E_0)$ with degree $u(2^e 2)$
 - Compute the 2D-kernel: $\langle ([u]P, \theta(P)), ([u]Q, \theta(Q)) \rangle$
 - Compute the isogeny from this kernel: $\Phi: E_0 \times E_0 \to E \times E'$
 - Evaluate $\Phi(P,0)$ and $\Phi(Q,0)$ to obtain the image of $\varphi: E_0 \to E$
 - Set $I = \mathcal{O}\theta + \mathcal{O}u$

• Output: the image of the basis under the action of an isogeny of degree u and the codomain

Ideal To Isogeny

- More complicated!
- Rough Idea:
 - Given an ideal, compute the action of the corresponding isogeny on $E[2^e]$
 - ullet Requires two calls to fixed degree isogeny built from the basis of I
 - Output allows the construction of a third 2D isogeny
 - We evaluate this to get the action desired

Computing the response is still complex



Is SQIsign2D Secure

SQIsign2D Security Assumptions **Key Recovery**

- As with preceding work of SQIsign*, key recovery requires solving the endomorphism ring problem
- We're fairly confident that this is hard
- Much more fundamentally hard than SIDH
- There's always the horror of maths to keep us worried

SQIsign2D Security Assumptions Knowledge Soundness

A nice SQIsign trait is the high soundness of the identification protocol
This means we only need to run the identification protocol a single time
Proof of soundness comes from the degree of the challenge isogeny
This is easy to control and set up for optimal parameters

SQIsign2D Security Assumptions Zero Knowledge

- Proof that the protocol has the zero-knowledge property
- Uses a simulator producing transcripts indistinguishable from a honest run of the protocol
- This simulator runs in polynomial time if it has access to an oracle producing random isogenies
- Much of our work is based on the previous work of SQIsignHD
- Things work nicely in dimension two because of Fixed Degree Isogeny

SQIsign2D Security Assumptions The Oracle known as FIDIO

A fixed degree isogeny oracle (FIDIO) is an oracle random isogeny $\varphi: E \to E'$ (in efficient representation) with domain E and degree N.

taking as input a supersingular elliptic curve E defined over \mathbb{F}_{p^2} and an integer N, and outputs a uniformly



SQIsign vs The World

SQIsign2D is a significant improvement						
	SQlsign			SQlsign2D		
	Keygen	Sign	Verify	Keygen	Sign	Verify
NISTI	2,800	4,600	93	120	290	25
NIST III	21,300	39,000	641	440	1,040	98
NIST V	91,600	165,000	2,080	1,070	2,490	247

Results measured in 10⁶ cycles (Intel Ice Lake, 2Ghz)



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gnificant improvement						
		SQlsign2[)			
/erify	Keygen	Sign	Verify			
39	60	160	9			
	170	460	29			
	360	940	62			

Keygen is up to 254x faster than SQlsign

SQlsign			SQlsign2D		
Keygen	Sign	Verify	Keygen	Sign	Verify
1,700	2,400	39	28x	160	9
21,300	39,000	641	125x	460	29
91,600	165,000	2,080	254x	940	62
	1,700 21,300	Keygen Sign 1,700 2,400 21,300 39,000	Keygen Sign Verify 1,700 2,400 39 21,300 39,000 641	Keygen Sign Verify Keygen 1,700 2,400 39 28x 21,300 39,000 641 125x	Keygen Sign Verify Keygen Sign 1,700 2,400 39 28x 160 21,300 39,000 641 125x 460

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NIST III	21,300	39,000	641	170	85x	29	
NIST V	91,600	165,000	2,080	360	175x	62	

Signing is up to 85x faster than SQIsign

SQIsign2D is a significant improvement							
SQlsign			SQlsign2D				
Keygen	Sign	Verify	Keygen	Sign	Verify		
1,700	2,400	39	60	160	4 x		
21,300	39,000	641	170	460	22x		
91,600	165,000	2,080	360	940	34 x		
	Keygen 1,700 21,300	SQIsign Keygen Sign 1,700 2,400 21,300 39,000	SQIsign Keygen Sign Verify 1,700 2,400 39 21,300 39,000 641	SQIsign SQ Keygen Sign Verify Keygen Sign 1,700 2,400 39 60 60 21,300 39,000 641 170 170	SQIsign SQIsign21 Keygen Sign Verify Keygen Sign 1,700 2,400 39 60 160 21,300 39,000 641 170 460		

Verification is up to 34x faster than SQIsign

	SQIsign2D gets close to "classical" sizes						
	ECDSA		ML-DSA		SQlsign2D		
	PK	Sig	PK	Sig	PK	Sig	
NISTI	32	64	1,312 2	,420	66	148	
NIST III	48	96	1,952 3	,309	98	222	
NIST V	64	128	2,592 4	,627	130	294	

	SQIsign2D gets close to "classical" sizes						
	ECDSA		Fal	Falcon		SQlsign2D	
	PK	Sig	PK	Sig	PK	Sig	
NISTI	32	64	897	666	66	148	
NIST III	48	96			98	222	
NIST V	64	128	1,793	1,280	130	294	

SQIsign2D is still magnitudes slower						
	ML-DSA	(Dilithium)	SQIsi	gn2D		
	Sign	Verify	Sign	Verify		
NISTI	333	118	160,000	9,000		
NIST III	529	179	460,000	29,000		
NIST V	642	279	940,000	62,000		

Results measured in 10³ cycles

SQIsign2D is still magnitudes slower						
	ML-DSA	(Dilithium)	SQIS	sign2D		
	Sign	Verify	Sign	Verify		
NIST I	333	118	480x	9,000		
NIST III	529	179	870x	29,000		
NIST V	642	279	1464x	62,000		

Results measured in 10³ cycles

SQIsign2D is still magnitudes slower						
	ML-DSA	(Dilithium)	SQIsi	gn2D		
	Sign	Verify	Sign	Verify		
NIST I	333	118	160,000	76x		
NIST III	529	179	460,000	162x		
NIST V	642	279	940,000	222x		

Results measured in 10³ cycles

What should we be working on?

Future Work What's Next?

- Cryptographic implementations • The current implementation has no side-channel protection
 - Constant time SQIsign is an open-problem!
- Efficient Implementations
 - Now the isogenies are faster, aspects of quaternions appear slow
 - How much more speed can we get by throwing every trick \mathbb{F}_{n^2} at the protocol?

Future Work What's Next?

• Dimension two isogenies are still the bottleneck • Can we find (4,4)-isogenies to improve performance • Can we make size/speed trade-offs to improve performance • Generalised Deuring for dimension two: • Long term goal, but would halve the characteristic • Convince masters students to take this as a PhD problem!

